Engineering Notes

Special Perturbations – KS with Time as the Independent Variable

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AN important subfunction of a spacecraft navigation system is orbit prediction. In the past, this has been done by integration of the equations of motion in Cartesian coordinates (Cowell's method) or calculating perturbations to a set of Keplerian or equivalent elements.[1] Orbit prediction based on perturbation of orbital elements yields the fastest answers and greatest accuracy at some cost in simplicity of method.

The elements of the orbit need not be the standard Keplerian elements, provided they are slowly varying and completely specify the state of the vehicle. The method of Kustaanheimo and Stiefel (KS)[2] provides such a set of elements by introducing a new independent variable called fictitious time and transforming to a four-dimensional coordinate system. In the new system, a set of equations is obtained which has the form of a set of perturbed harmonic oscillators. The regularized elements of the KS technique are the eigenstates of the unperturbed, linear system.

The KS method provides the most stable, accurate technique for orbit prediction presently available. As KS is presented in the literature, however, time is treated as a dependent variable in the calculation. This may be inconvenient in some applications, and the task of predicting a set of elements for a particular time is somewhat complicated by the need to interpolate among elements generated at successive times bracketing the desired time. A KS trajectory integrator could not be readily substituted into an existing spacecraft navigation system without redefining the step size input to the integrator, which might require significant modifications throughout the system to assure compatibility. These difficulties can be overcome and most of the accuracy and stability of the KS method for low-eccentricity orbits can be retained by calculating perturbations of the regularized elements in time rather than "fictitious time." The purpose of this Note is to present a set of differential equations for the regularized elements as a function of time.

Regularized Elements

The equation of motion for a satellite can be written in the form

$$\ddot{x} + \mu x/r^3 = -\partial V/\partial x + P \tag{1}$$

where $-\partial V/\partial x$ and P represent conservative and nonconservative perturbing accelerations, respectively. The physical vector x is three dimensional; however, no difficulty is caused by extending the dimension of x to four if the fourth component is defined to always be zero. Equation (1) is transformed to the coordinate system defined by the four-vector u via the transformation

$$x = L(u)u \tag{2}$$

where

$$L = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix} \tag{3}$$

and time, t, is replaced by the parameter s given by

$$dt/ds = r \tag{4}$$

The resulting set of transformed equations is solved by variation of parameters and yields the following set of differential equations for the new elements α, β, ω, and τ.

$$\omega^* = Q \tag{5}$$

$$\tau^* = T \tag{6}$$

$$\alpha^* = S \sin(E/2) \tag{7}$$

$$\beta^* = -S \cos(E/2) \tag{8}$$

The asterisk indicates differentiation with respect to the generalized eccentric anomaly E, which is related to the parameter s by

$$dE/ds = 2\omega \tag{9}$$

ω being the non-Keplerian orbital frequency

$$\omega = \left[\frac{1}{2}\left(\mu/r - v^2/2 - V\right)\right]^{1/2} \tag{10}$$

Additionally

$$Q = -(r/8\omega^2)\partial V/\partial t - (1/2\omega)u^* \cdot L^T P \tag{11}$$

$$T = (1/8\omega^3)(\mu - 2rV) - (r/16\omega^3)u \cdot (\partial V/\partial u - 2L^T P) - (2/\omega^2)\omega^* u \cdot u^* \tag{12}$$

$$S = \frac{1}{2\omega^2}\left[\frac{V}{2}\mu + r/4(\partial V/\partial u - 2L^T P)\right] + \frac{2}{\omega}\omega^* u^* \tag{13}$$

The position vector in the parametric space of u is related to the elements by

$$u = \alpha \cos(E/2) + \beta \sin(E/2) \tag{14}$$

The physical time is given by

$$t = \tau - u \cdot u^*/\omega \tag{15}$$

Equations (5-8) can be integrated to find the elements α and β as a function of eccentric anomaly. The u vector is obtained from Eq. (14) and time from Eq. (15), then the physical position vector is found via Eq. (2). The velocity is found using

$$u^* = -\tfrac{1}{2}\alpha \sin(E/2) + \tfrac{1}{2}\beta \cos(E/2) \tag{16}$$

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and

$$v = 4\omega\, L(u)\, u^*/r \tag{17}$$

Equations (5-8) may be easily changed so that derivatives are taken with respect to time instead of E. From Eqs. (4) and (9), it follows that

$$\dot{E} = 2\omega/r \tag{18}$$

and the time derivative of any element f can be written

$$\dot{f} = f^* \dot{E} \tag{19}$$

but a way of calculating E as a function of time must be provided. Equation (18) provides the answer. Equation (18) is equivalent to

$$t^* = r/2\omega \tag{20}$$

where

$$r = u \cdot u \tag{21}$$

as follows from Eqs. (2) and (3). From Eqs. (14, 20, and 21) it follows

$$t^* = \frac{1}{2\omega}\left(\frac{\alpha^2+\beta^2}{2} + \frac{\alpha^2-\beta^2}{2}\cos E + \alpha\cdot\beta \sin E\right) \tag{22}$$

Solving Eq. (22) by variation of parameters yields

$$t = t_0 + \frac{1}{2\omega}\left[\frac{\alpha^2+\beta^2}{2}E + \frac{\alpha^2-\beta^2}{2}\sin E + \alpha\cdot\beta(1-\cos E)\right] \tag{23}$$

where t_0 is a new element which may be called the initial time element. Equation (23) is essentially Kepler's equation, although the elements which appear in the equation are not identically Keplerian except in the absence of a perturbing potential.

In order that Eqs. (22) and (23) be simultaneously satisfied, the initial time must have a derivative given by

$$t_0^* = (t-t_0)\frac{\omega^*}{\omega} - \frac{1}{2\omega}\Big[\alpha\cdot\alpha^*(E+\sin E) + \beta\cdot\beta^*(E-\sin E) + (\alpha^*\cdot\beta + \alpha\cdot\beta^*)(1-\cos E)\Big] \tag{24}$$

Equations (5, 7, 8, and 24) can be expressed as derivatives with respect to time simply by multiplying each equation by $\dot{E}$. The result is

$$\dot{\omega} = Q\dot{E} \tag{25}$$

$$\dot{\alpha} = S\dot{E}\sin(E/2) \tag{26}$$

$$\dot{\beta} = -S\dot{E}\cos(E/2) \tag{27}$$

$$\dot{t}_0 = (t-t_0)\frac{\dot{\omega}}{\omega} - \frac{1}{2\omega}\Big[\alpha\cdot\dot{\alpha}(E+\sin E) + \beta\cdot\dot{\beta}(E-\sin E) + (\dot{\alpha}\cdot\beta + \alpha\cdot\dot{\beta})(1-\cos E)\Big] \tag{28}$$

where

$$Q\dot{E} = -\frac{1}{4\omega}\frac{\partial V}{\partial t} - \frac{1}{r}u^*\cdot L^T P \tag{29}$$

$$S\dot{E} = \frac{1}{\omega r}\left[\frac{V}{2}\mu + \frac{r}{4}\left(\frac{\partial V}{\partial u} - 2L^T P\right)\right] + \frac{2}{\omega}\dot{\omega}u^* \tag{30}$$

The differential equations of the KS elements are now written in terms of derivatives with respect to time and may be integrated using standard techniques. Kepler's equation [Eq. (23)] is solved at each integration step to obtain the value of E corresponding to the specified time, t.

A price is paid for writing the differential equations as functions of time, however. The time step is no longer internally controlled by the regularization, but must be externally controlled. The original set of equations could integrate through collisions, (r very small) but the appearance of terms proportional to $1/r$ precludes this with the new set. Even in this "deregularized" form, however, the KS method still provides a more stable set of equations than a variation of parameters technique based on Keplerian elements. Computer experiments have verified that the non-Keplerian elements of KS are more slowly varying than Keplerian elements and that feature is preserved in the time dependent set.

References

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[2] Stiefel, E. L. and Scheifele, G., *Linear and Regular Celestial Mechanics*, Springer-Verlag, New York, 1971., pp. 83-92.

Gravitationally Stabilized Satellite Solar Power Station in Orbit

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I. Introduction

THE possibility of collecting large amounts of solar energy in space has been studied for a number of years.[1,2] The principal advantage of such schemes is the availability of practically inexhaustable amounts of energy which can be harnessed away from the Earth's active environment. Large structures can be constructed and placed in the synchronous equatorial orbit to collect and transmit solar energy (via microwave) to selected locations on the Earth. No energy storage devices to compensate for the day-night cycles or cloundy weather are required. It is estimated that a 5 × 12 km solar array can, for example, deliver 5000 MW of power on the ground, assuming a solar array efficiency of 11.3%.[3]

The present study examines an alternative approach employing a larger number of smaller solar collectors attached to a gravitationally stabilized cable in orbit. The problems associated with the construction and active attitude control of very large structures are thus alleviated. A schematic drawing of this concept is shown in Fig. 1 for a 5-GW solar photovoltaic system consisting of fifty-two 750 × 1500-m panels attached to a 57.75 km tapered cable with a 0.83-km-diam microwave (S-band, 2.45 GHz) transmitting antenna. The total solar array area is 58.5 km^2. The total mass has been estimated to be on the order of 18×10^6 kg. The slipring-connected solar segments are assumed actively controlled within a few degrees to the sun while the long axis of the array is Earth-pointing. A roll oscillation (about the satellite orbital velocity vector) is induced to prevent mutual shadowing of the solar segments when the sun is directly above the array.

The microwave antenna can be steered electronically with a scan angle capability of up to 60 deg thus requiring little or no active attitude control. The accuracy of pointing can be determined to 0.12 mrad. The elimination of sliprings and

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